UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH1402**
ASSESSMENT PATTERN	:	**MATH1402A**
MODULE NAME	:	**Mathematical Methods 2**
DATE	:	**11-May-09**
TIME	:	**10:00**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2008/09-MATH1402A-001-EXAM-203

TURN OVER

All questions may be answered, but only marks obtained on the best **four** *questions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. a) Write down the formula of the linear Taylor's approximation of a function $f(x, y)$ near a point (x_0, y_0).
   What can we say about the rate of this approximation?

   b) Show that the tangent plane to the graph of a function $f(x, y)$ at a point (x_0, y_0, z_0), $z_0 = f(x_0, y_0)$ is a horizontal plane if and only if $\nabla f(x_0, y_0) = \mathbf{0}$.

   c) For the function $f(x, y) = x + e^{xy}$ find the equation of its tangent plane at the point $(1, 0, 2)$.

   d) For the function f from Part (c), find a vector $\mathbf{u} \neq \mathbf{0}$ which is orthogonal to $\nabla f(1, 0)$.

2. a) Let R be a region on the $xy-$ plane defined by

$$x^2 + y^2 \leq 1, \ x \geq 0, y \leq 0.$$

   Find the integral

$$\iint_R e^{(x^2+y^2)} x^2 dx dy.$$

   b) Let the surface S be the graph of the function $f(x, y) = \exp(x + y)$, where (x, y) satisfy

$$|x| + |y| \leq 2.$$

   Find the surface integral

$$\iint_S z^2 \, dS.$$

   [Hint: Use the change of variables: $u = x + y$, $v = x - y$.]

3. a) State the Divergence Theorem carefully.

   b) Let D be a cylinder,

$$x^2 + y^2 \leq 1, \quad 0 \leq z \leq 2.$$

   Let $\mathbf{F}$ be a vector field

$$\mathbf{F}(x, y, z) = (1 - a^2)x^3\mathbf{i} + (1 - a^2)y^3\mathbf{j} + (x^2 + y^2)z\mathbf{k},$$

   where a is a real number. Find the flux of $\mathbf{F}$ through S, where S is the surface surrounding D.

c) Let $\mathbf{F}$ and S be as in Part (b). Find the values of a when the flux is equal to 0.

d) Use the Divergence Theorem to prove the First Green's identity:

$$\iiint_V \left(f\,\nabla^2 g + \nabla f \cdot \nabla g\right)\,dxdydz = \iint_S f\frac{\partial g}{\partial \mathbf{n}}\,dS.$$

Here $f(x,y,z)$, $g(x,y,z)$ are smooth functions in a bounded domain $V \subset \mathcal{R}^3$, S is a smooth surface surrounding V and $\mathbf{n}$ is an outward-looking unit normal to V.

4. a) State Stoke's Theorem carefully.

b) Verify Stoke's Theorem for the vector field

$$\mathbf{F}(x,y,z) = y\mathbf{i} + 2z\mathbf{j} + xz\mathbf{k}$$

and the surface S defined by

$$x^2 + y^2 + z^2 = 25, \quad z \geq 4.$$

5. a) State Green's Theorem in the plane carefully.

b) Sketch the closed curve C which is described as follows: Begin at point $(0,0)$ and go to the point $(\pi, 0)$ along the straight line. Then go back to $(0,0)$ along the curve described by the equation $y = \sin(x)$. This description also gives a correct orientation of C.

c) Let

$$\mathbf{F}(x,y) = y\mathbf{i} + (x^2 y + \exp(y^2))\mathbf{j}.$$

Use Green's Theorem to calculate the circulation of $\mathbf{F}$ around C.

6. a) Describe the necessary and sufficient conditions such that , for any $\mathbf{X_0}$, $\mathbf{X_1} \in \mathcal{R}^3$, the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of the choice of C and depends only on $\mathbf{X_0}$ and $\mathbf{X_1}$.

b) Let

$$\mathbf{F} = \frac{2xz}{1+x^2z}\mathbf{i} + y\mathbf{j} + \frac{x^2}{1+x^2z}\mathbf{k}.$$

Show that $\nabla \times \mathbf{F} = \mathbf{0}$ and find a potential function for $\mathbf{F}$.

c) Let

$$\mathbf{G} = \frac{2xz}{1+x^2z}\mathbf{i} + (x+y)\mathbf{j} + \frac{x^2}{1+x^2z}\mathbf{k}.$$

Let C be a unit circle centered at $\mathbf{O} = (0,0,0)$ lying in the plane $y = z$. Find

$$\oint_C \mathbf{G} \cdot d\mathbf{r}.$$

[Hint: Use Part (b).]